Ionization fronts in negative corona discharges

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We use a hydrodynamic minimal streamer model to study negative corona discharge. By reformulating the model in terms of a quantity called a shielding factor, we deduce laws for the evolution in time of both the radius and intensity of the ionization fronts. We also compute the evolution of the front thickness under the conditions for which it diffuses due to the geometry of the problem and show its self-similar character.

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A common feature in transient discharges preceding dielectric breakdown is the creation of a nonequilibrium plasma through the propagation of a nonlinear ionization wave into a previously nonionized region. Modern concepts of pattern formation which have been already applied in different contexts (see, e.g, [1,2]) have also been used in order to gain new analytical insight into this old problem [3].

When a sufficiently high voltage is suddenly applied to a medium with low or vanishing conductivity, extending fingers of ionized matter develop. They are called streamers and are ubiquitous in nature and in technology [4,5]. A minimal streamer model, consisting of a fluid approximation with local field-dependent impact ionization in a nonattaching gas like argon or nitrogen, has been used to study the basics of streamer dynamics [3,6–9].

We consider in this paper the evolution of negative ionization fronts in a gas under the influence of a nonuniform external electric field. The field is created by a potential difference V_0 applied between a pair of electrodes. The geometry of the electrodes determines the nonuniformity of the electric field. A discharge develops in the high-field region near the sharper electrode, and it spreads out towards the other electrode. This type of discharge is called a corona. It is a negative corona discharge when the electrode with the strongest curvature is connected to the negative terminal of the power supply. We will consider this case.

The so-called minimum model [3,8–10] consists in the following dimensionless set of equations:

$$\frac{\partial n_e}{\partial t} - \nabla \cdot \mathbf{j} = n_e f(\mathbf{E}), \tag{1}$$

$$\frac{\partial n_i}{\partial t} = n_e f(\mathbf{E}), \tag{2}$$

$$\nabla \cdot \mathbf{E} = n_i - n_e. \tag{3}$$

Equation (1) describes the rate of change of the local dimensionless electron density n_e. It is equal to the divergence of the local electron current density $\mathbf{j}$ plus a source term $n_e f(\mathbf{E})$ representing the generation of electron-ion pairs due to the impact of accelerated electrons with neutral gas molecules. The value of $f(\mathbf{E})$ is given by the Townsend approximation

$$f(\mathbf{E}) = E \exp(-1/E), \tag{4}$$

where E is the modulus of the local electric field $\mathbf{E}$. In Eq. (2) we consider that the rate of change of the ion density n_i is equal to the source term due to impact, since we take the ion current density to be negligible in a first approximation (the speed of ions is typically much smaller than that of electrons). The local value of the electron current density is specified as

$$\mathbf{j} = n_e \mathbf{E} + D \nabla n_e, \tag{5}$$

using Ohm's law in the first term and considering diffusion effects in the second one. Note that this expression does not include the effect of the magnetic field created by the motion of electrons, as it is supposed that their speed is much smaller than the speed of light. Expression (3) is Poisson's equation, coupling the electric field to the charge densities.

Since our primary goal in this paper is to address the effects of curvature in front propagation, we will neglect diffusion effects [9]. That allows us to reduce the set of equations (1)–(3) to a simpler form in order to give analytical results for the evolution of the ionization fronts. From Eqs. (1), (2), and (5), with $D=0$, we obtain

$$\frac{\partial}{\partial t}(n_i - n_e) + \nabla \cdot (n_e \mathbf{E}) = 0, \tag{6}$$

and from Eq. (3), taking the time derivative,

$$\nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) = \frac{\partial}{\partial t}(n_i - n_e). \tag{7}$$

Equations (6) and (7) give then

$$\nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} + n_e \mathbf{E} \right) = 0. \tag{8}$$

The term inside brackets in Eq. (8) is, due to Maxwell equations, proportional to the curl of the magnetic field in the gas. As it is supposed that the magnetic field is negligible, we can take it equal to zero and integrate in time, yielding

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r}) \exp\left(- \int_0^t d\tau n_e(\mathbf{r},\tau) \right), \tag{9}$$

which gives the local electric field $\mathbf{E}$ in terms of the initial electric field $\mathbf{E}_0$ and the electron density n_e integrated in time. Equation (9) motivates the definition of the quantity

$$u(\mathbf{r},t) = \exp\left(- \int_0^t d\tau n_e(\mathbf{r},\tau) \right). \tag{10}$$

If this quantity is completely determined in a particular problem, then using Eqs. (9), (10), and (3) all the physical fields can be obtained through the expressions

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})u(\mathbf{r},t), \tag{11}$$

$$n_e(\mathbf{r},t) = - \frac{1}{u(\mathbf{r},t)} \frac{\partial u(\mathbf{r},t)}{\partial t}, \tag{12}$$

$$n_i(\mathbf{r},t) = - \frac{1}{u(\mathbf{r},t)} \frac{\partial u(\mathbf{r},t)}{\partial t} + \boldsymbol{\nabla} \cdot [\mathbf{E}_0(\mathbf{r})u(\mathbf{r},t)], \tag{13}$$

in which the initial condition $\mathbf{E}_0$ for the electric field should be known. Equation (11) reveals clearly the role played by the function $u(\mathbf{r},t)$ as a factor modulating the electric field $\mathbf{E}$ at any time. The electronic density is positive, so $u(\mathbf{r},t)$ decays, damping the electric field. For this reason we call it the *shielding factor*. The shielding factor determines a screening length which changes with time: a kind of Debye length which moves with the front, leaving behind a neutral plasma.

The problem is thus reduced to finding equations and conditions for the shielding factor $u(\mathbf{r},t)$ from equations and conditions for the physical quantities $\mathbf{E}$, n_e, and n_i. Substituting Eqs. (11)–(13) into the model equations (1)–(3), after some algebraic manipulations and integrating once in time, the evolution of $u(\mathbf{r},t)$ is given by

$$\frac{1}{u}\frac{\partial u}{\partial t} = \boldsymbol{\nabla} \cdot (\mathbf{E}_0 u) - n_{i0}(\mathbf{r}) - \int_{E_0 u}^{E_0} \exp\left(\frac{-1}{s}\right) ds, \tag{14}$$

$$u(\mathbf{r},0) = u_0(\mathbf{r}) = 1, \tag{15}$$

where E_0 is the modulus of $\mathbf{E}_0$ and n_{i0} is the initial ion density. Boundary conditions should be imposed depending on the particular physical situation.

In what follows we will consider a typical corona geometry: two spherical plates with internal radius R_0 and $R_1 \gg R_0$, respectively. An electric potential difference V_0 is applied to these plates, so that $V(R_1) - V(R_0) = V_0 > 0$. The initial seed of ionization is taken to be neutral so that

$$n_{e0}(r) = n_{i0}(r) = \rho_0(r). \tag{16}$$

We consider the evolution of negative ionization fronts towards the positive plate at $r = R_1$. The initial electric field $\mathbf{E}_0(r)$ between the plates is

$$\mathbf{E}_0(r) = - \frac{C}{r^2}\mathbf{u}_r, \quad C = V_0 \frac{R_0 R_1}{R_1 - R_0}. \tag{17}$$

We substitute Eq. (17) into Eq. (14) and change the spatial variable r to

$$x = \frac{r^3}{3C}, \tag{18}$$

so that the evolution for the screening factor takes the form

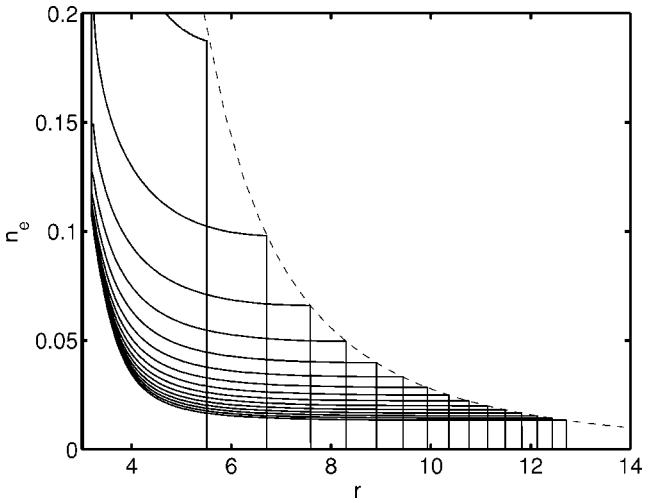

FIG. 1. The shock-wave development at regular intervals of time when the initial charge distribution is well localized. The amplitude of the front calculated analytically is plotted as the dashed line.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = - u\rho_0(x) - u \int_{u(C/9x^2)^{1/3}}^{(C/9x^2)^{1/3}} \exp\left(\frac{-1}{s}\right) ds. \tag{19}$$

Equation (19) governing the behavior of the screening factor is a Burgers-type equation, where $\rho_0(x)$ is the initial distribution of charge. The condition for the initial value of the screening factor is, by Eq. (15), $u(x,0) = 1$. Following the usual procedure of resolution of the Burgers equation we can integrate Eq. (19) along the characteristics $x_c(t)$ defined by

$$\frac{dx_c(t)}{dt} = u(x_c(t),t), \tag{20}$$

transforming Eq. (19) into an ordinary differential equation.

First the case of sufficiently localized initial conditions is considered. More specifically, the initial electron density strictly vanishes beyond a certain point. Under similar conditions, the existence of shock fronts with constant velocity has been predicted for the simpler planar geometry [3].

Taking a homogeneous thin layer of width $\delta \ll (R_1 - R_0)$ from $r = R_0$ to $r = R_0 + \delta$, the initial charge distribution is then

$$n_{e0}(r) = n_{i0}(r) = \rho_0, \quad R_0 < r < R_0 + \delta,$$

$$n_{e0}(r) = n_{i0}(r) = 0, \quad R_0 + \delta < r < R_1. \tag{21}$$

In Fig. 1 we show the electron density distribution n_e which corresponds to some arbitrary choice of parameters ρ_0, δ, R_0, R_1, and V_0. The electron density has been calculated using expression (12) and plotted as a function of r at different times t. There appears a sharp shock with decaying amplitude, separating the region with charge and the region without charge.

From these numerical data, we can measure the velocity of propagation of such a front. In Fig. 2 the position of the shock r_f is plotted as a function of time. The velocity of

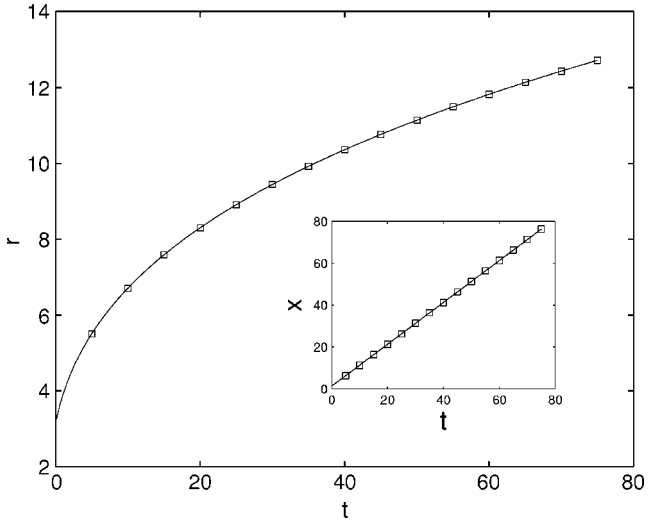

FIG. 2. Position of the shock front versus time. Squares are numerical results, and the solid lines are the theoretical predictions. In the inset the position in the scaled variable x is shown. The linear dependence in this variable as predicted in Eq. (26) can be observed.

propagation is clearly not constant. However, if we plot the position of the front in terms of x, one can observe the linear relation (see the inset in Fig. 2)

$$x_f(t) = t + x_0, \tag{22}$$

which implies, in terms of the original variable r, an asymptotic behavior

$$r_f(t) \sim (3C)^{1/3} t^{1/3} \tag{23}$$

for the position of the front.

Remarkably we can deduce expressions for both the amplitude and propagation velocity of the shock in explicit analytical form. In order to do that, we write locally near the front the solution as

$$u(x,t) = 1 - a(t)\varphi(\xi), \quad \xi = x - x_f(t). \tag{24}$$

We substitute this ansatz into Eq. (19), and since the integral term is small when $x \gg 1$, we get

$$a(t)\varphi'(\xi) - a(t)\varphi'(\xi)x_f'(t) - a'(t)\varphi(\xi) + a^2(t)\varphi(\xi)\varphi'(\xi) \approx 0, \tag{25}$$

implying that

$$x_f(t) = t + x_0, \tag{26}$$

$$a(t) = \frac{\beta}{(t + t_0)}, \tag{27}$$

$$\varphi(\xi) = \beta^{-1}(x - t - x_0), \tag{28}$$

where β is an arbitrary constant to be fixed from initial conditions. Equation (26) proves that the position of the shock front follows the law (23) for spherical geometry.

From Eq. (12), the electron density reads

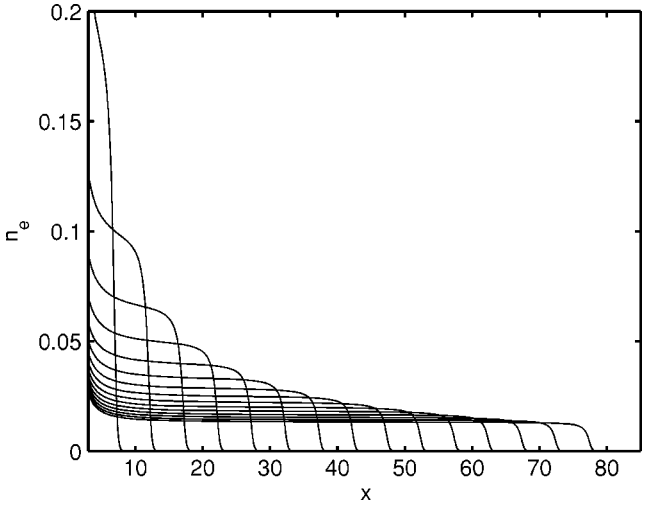

FIG. 3. Front development when the initial condition is not localized, plotted at regular intervals of time.

$$n_e(x,t) \approx \begin{cases} \dfrac{1}{t + t_0} \dfrac{1 + (x - t - x_0)/(t + t_0)}{1 - (x - t - x_0)/(t + t_0)}, & x \leqslant t + x_0, \\ 0, & x > t + x_0, \end{cases} \tag{29}$$

which implies that the amplitude of the front decays as

$$n_e(x_f(t),t) = \frac{1}{t + t_0}. \tag{30}$$

In Fig. 1 the analytical curve (30) has been plotted as the dashed line, showing excellent agreement with the numerical data.

We want to conclude this paper with a brief discussion of the case in which the initial charge distribution is not localized as, for instance, one such that

$$n_{e0}(x) = n_{i0}(x) = \rho_0(x) \sim e^{-\lambda x}, \quad x \gg 1. \tag{31}$$

For the planar case it was predicted in [3] that the front propagated with a constant velocity, although no shock front would develop unless the decay is sufficiently fast.

As we did above, we solve the problem numerically assuming spherical symmetry, so x is related to the radial coordinate r by Eq. (18). In Fig. 3 the solution for the electron density is shown. The shock front does not appear in this case. Instead, a front with increasing thickness propagates. In the scaled variable x, we have checked that the center of this front moves with constant velocity as the shock front does. These facts are apparent from the figure.

Using scaling arguments, it can be shown that the asymptotic local behavior near the front can be described in the self-similar form

$$n_e(x,t) \approx \frac{1}{t} f\left(\frac{\xi}{\delta_\lambda t}\right), \tag{32}$$

where $\xi = x - t$, f is some universal self-similar profile and δ_λ is a constant measuring the front thickness in rescaled units. Its value depends on the physical parameters and initial con-

ditions. Hence the front presents a typical thickness

$$\xi_c \approx \delta_\lambda t. \tag{33}$$

The fact that, even neglecting diffusion, the front spreads out linearly in time is a remarkable feature of the curved geometry considered here. For this reason it can be termed as *geometrical diffusion*. In Fig. 4 we have plotted the numerical solutions rescaled according to Eq. (32) showing a clear convergence towards a universal profile.

The principal results and contributions from the work presented in this paper can be summarized as follows. First we have introduced the shielding factor as the factor damping the electric field in nonequilibrium electric discharges when the magnetic field can be considered negligible, Eq. (10). This factor defines a characteristic length analogous to Debye length for stationary discharges. The physics contained in the minimum model for streamer discharges can be reduced to the study of the evolution of the shielding factor. We have derived the equation which governs its evolution, Eq. (14), for a gas like nitrogen or argon without taking into account the diffusion of charged species and the processes of photoionization.

Then we have considered the case of a negative corona discharge with spherical symmetry. In this case, the discharge takes place in a nonhomogeneous electric field and the equation for the shielding factor turns out to be a Burgers one. We have extended the results of planar fronts to this case where the geometry is curved. Depending on the initial conditions for the charge distribution, one might have negative shocks or spreading fronts. In both cases, the amplitude decreases in time and the propagation velocity follows a power law. In the case of spreading fronts we have proved the appearance of diffusion-type phenomena due to purely geometrical effects.

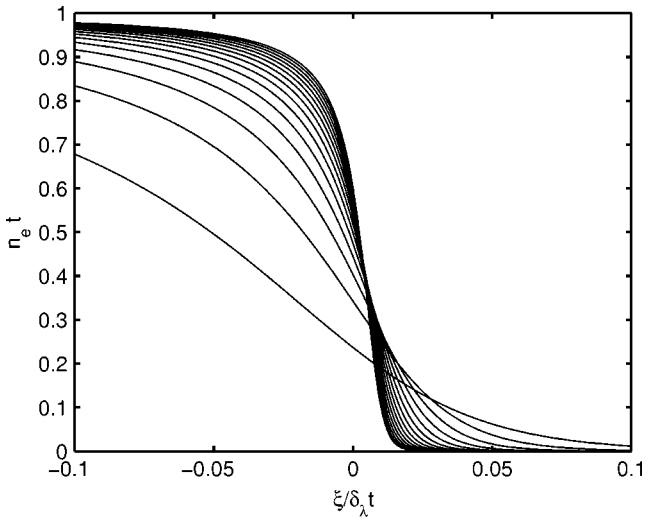

FIG. 4. Front profiles rescaled according to the self-similar law given by expression (32). The profiles converge asymptotically to function f in that expression.

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